LIPSCHITZ EXTENSION OF MULTIPLE BANACH-VALUED FUNCTIONS IN THE SENSE OF ALMGREN

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Abstract. A multiple-valued function $f : X \to \mathbf{Q}_Q(Y)$ is essentially a rule assigning Q unordered and non necessarily distinct elements of Y to each element of X. We study the Lipschitz extension problem in this context by using two general Lipschitz extension theorems recently proved by U. Lang and T. Schlichenmaier. We prove that the pair $\left(X, \mathbf{Q}_Q(Y)\right)$ has the Lipschitz extension property if Y is a Banach space and X is a metric space with a finite Nagata dimension. We also show that $\mathbf{Q}_Q(Y)$ is an absolute Lipschitz retract if Y is a finite algebraic dimensional Banach space.

1. Introduction

A multiple-valued function in the sense of Almgren is a map of the form $f : X \to \mathbf{Q}_Q(Y)$ where X and (Y, d) are metric spaces. The particular target space is defined by

$$\mathbf{Q}_Q(Y) = \left\{ \sum_{i=1}^{Q} [\![x_i]\!] \;:\; x_i \in Y \quad \text{for } i = 1, \ldots, Q \right\}$$

where $[\![x_i]\!]$ denotes the Dirac measure at x_i and it is equipped with the metric

$$\mathcal{S}\left(\sum_{i=1}^{Q}[\![x_i]\!], \sum_{j=1}^{Q}[\![y_j]\!]\right) = \min\left\{ \max_{i=1,\ldots,Q} d\left(x_i, y_{\sigma(i)}\right) \;:\; \sigma \text{ is a permutation of } \{1, \ldots, Q\} \right\}.$$

Consequently a multiple-valued function $f : X \to \mathbf{Q}_Q(Y)$ is essentially a rule assigning Q unordered and non necessarily distinct elements of Y to each element of X. Such maps are studied in complex analysis (see appendix 5 in [10]). Indeed in complex function theory one often speaks of the "two-valued function $f(z) = z^{1/2}$ ". This can be considered as a function from $\mathbb{C}$ to $\mathbf{Q}_2(\mathbb{C})$.

In his big regularity paper [1], F. J. Almgren introduced $\mathbf{Q}_Q(\mathbb{R}^n)$-valued functions to tackle the problem of estimating the size of the singular set of mass-minimizing integral currents (see [2] for a summary). Almgren's multiple-valued functions are a fundamental tool for understanding geometric variational problems in codimension higher than 1. The success of Almgren's regularity theory raises the need of further studying multiple-valued functions.

The Lipschitz extension problem asks for conditions on a pair of metric spaces X, Y such that every Lipschitz Y-valued function defined on a subset of X can be extended to all of X with only a bounded multiplicative loss in the Lipschitz constant. More precisely the pair (X, Y) is said to have the Lipschitz extension property if there exists a constant $\lambda \geq 1$ such that for every subset $A \subset X$, every Lipschitz function $f : A \to Y$ can be extended to a Lipschitz function $F : X \to Y$ with $\operatorname{Lip}(F) \leq \lambda \operatorname{Lip}(f)$. A metric space Y is said to be an absolute Lipschitz retract if for every metric space X, the pair (X, Y) has the Lipschitz extension property (see chapter 1 in [3] for equivalent definitions). This problem dates back to the work of Kirszbraun and Whitney in the 1930's, and has been extensively investigated in the last two decades (see [8] and [9] for several recent breakthroughs).

In the present paper, we will be interested in the Lipschitz extension of $\mathbf{Q}_Q(Y)$-valued functions when Y is a Banach space. In this context, an important remark is that a Lipschitz $\mathbf{Q}_Q(Y)$-valued

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function is much more than Q glued Lipschitz Y-valued functions. Indeed we noticed in [5] that the following Lipschitz $\mathbf{Q}_2(\mathbb{R}^2)$-valued function

$$f : \mathbf{S}^1 \subset \mathbb{R}^2 \to \mathbf{Q}_2(\mathbb{R}^2) : x = (\cos\theta, \sin\theta) \mapsto f(x) = \left[\!\!\left[\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2} \right) \right]\!\!\right] + \left[\!\!\left[\left(-\cos\frac{\theta}{2}, -\sin\frac{\theta}{2} \right) \right]\!\!\right]$$

doesn't split into two Lipschitz $\mathbb{R}^2$-valued branches. Consequently the Lipschitz extension problems for $\mathbf{Q}_Q(Y)$-valued functions and Y-valued functions are in general two distinct problems.

In [1], Almgren built an explicit bilipschitz correspondence between $\mathbf{Q}_Q(\mathbb{R}^n)$ and a Lipschitz retract denoted Q^* included in a Euclidean space. This construction and McShane-Whitney's Theorem (see 2.10.44 in [6]) clearly imply that $\mathbf{Q}_Q(\mathbb{R}^n)$ is an absolute Lipschitz retract. For multiple Banach-valued functions, such a bilipschitz correspondence is not available.

In a recent paper [8], U. Lang and T. Schlichenmaier obtained two general Lipschitz extension theorems involving a Lipschitz connectedness assumption on the target space and a bound on the Nagata dimension denoted $\dim_N$ below of either the source space or the target space:

Theorem 1.5 in [8]. Suppose that X, Y are metric spaces, $\dim_N(X) \leq n < \infty$, and Y is complete. If Y is Lipschitz $(n-1)$-connected, then the pair (X, Y) has the Lipschitz extension property.

Corollary 1.8 in [8]. Suppose that Y is a metric space with $\dim_N(Y) \leq n < \infty$. Then Y is an absolute Lipschitz retract if and only if Y is complete and Lipschitz n-connected.

In Section 2 we prove that $\mathbf{Q}_Q(Y)$ is complete in case Y is. We recall what is meant by Lipschitz connectedness and we prove that $\mathbf{Q}_Q(Y)$ enjoys this property when Y is a weakly convex geodesic space.

In Section 3 we define the Nagata dimension and gather a number of basic properties. We estimate the Nagata dimension of $\mathbf{Q}_Q(Y)$ in accordance with the Nagata dimension of Y. We also show that $\mathbf{Q}_Q(Y)$ has a finite Nagata dimension when Y is a finite algebraic dimensional Banach space.

We finally combine these results with Lang-Schlichenmaier's Theorems in order to prove Theorem 1.1 and Theorem 1.2.

Theorem 1.1. Suppose that X is a metric space, $\dim_N(X) < \infty$, and Y is a complete weakly convex geodesic space. Then the pair $(X, \mathbf{Q}_Q(Y))$ has the Lipschitz extension property. In particular, the pair $(X, \mathbf{Q}_Q(Y))$ has the Lipschitz extension property if Y is a Banach space.

Theorem 1.2. If Y is a complete weakly convex geodesic space with a finite Nagata dimension then $\mathbf{Q}_Q(Y)$ is an absolute Lipschitz retract. In particular, $\mathbf{Q}_Q(Y)$ is an absolute Lipschitz retract if Y is a Banach space with a finite algebraic dimension.

2. Completeness and Lipschitz connectedness of $\mathbf{Q}_Q(Y)$

For later use we note a simple fact related to the completeness property.

Lemma 2.1. If Y is a complete metric space, then $\mathbf{Q}_Q(Y)$ is complete.

Proof. Let $(x^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbf{Q}_Q(Y)$. It is enough to show that we can extract a converging subsequence. On the one hand, it is clear that we can extract a subsequence $(x^{n_l})_{l\in\mathbb{N}}$ such that $\mathcal{S}(x^{n_l}, x^{n_{l+1}}) < 1/2^l$ for all $l \in \mathbb{N}$. On the other hand, we can write $x^{n_l} = \sum_{i=1}^{Q} [\![x_i^{n_l}]\!]$ with $\mathcal{S}(x^{n_l}, x^{n_{l+1}}) = \max_{i=1,\dots,Q}\ d(x_i^{n_l}, x_i^{n_{l+1}})$ for all $l \in \mathbb{N}$. We obtain that $d(x_i^{n_l}, x_i^{n_{l+1}}) < 1/2^l$ hence $(x_i^{n_l})_{l\in\mathbb{N}}$ is a Cauchy sequence in Y for $i = 1, \dots, Q$. Since Y is complete, there exist $x_1, \dots, x_Q \in Y$ such that $x_i^{n_l} \to x_i$ as $l \to \infty$ for $i = 1, \dots, Q$. Consequently, the subsequence $(x^{n_l})_{l\in\mathbb{N}}$ tends to $\sum_{i=1}^{Q} [\![x_i]\!] \in \mathbf{Q}_Q(Y)$ as $l \to \infty$. $\square$

Recall that a topological space Y is said to be n-connected, for some integer $n \geq 0$, if for every $m \in \{0, 1, \dots, n\}$, every continuous map from $\mathbf{S}^m$ into Y admits a continuous extension to $\mathbf{B}^{m+1}$. Accordingly, we call a metric space Y Lipschitz n-connected if there is a constant $\lambda \geq 1$ such

that for every $m \in \{0, 1, \ldots, n\}$, every Lipschitz map $f : \mathbf{S}^m \to Y$ possesses a Lipschitz extension $F : \mathbf{B}^{m+1} \to Y$ with $\mathrm{Lip}(F) \le \lambda \,\mathrm{Lip}(f)$. Here $\mathbf{S}^m$ and $\mathbf{B}^{m+1}$ denote the unit sphere and closed ball in $\mathbb{R}^{m+1}$, equipped with the induced metric.

Let (Y, d) be a metric space. A geodesic joining $x \in Y$ to $y \in Y$ is a map c_{xy} from a closed interval $[0, l] \subset \mathbb{R}$ to Y such that $c_{xy}(0) = x$, $c_{xy}(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$ (in particular, $l = d(x, y)$). A geodesic bicombing on Y is an assignment of a geodesic $c_{xy} : [0, d(x, y)] \to Y$ from x to y for every pair $(x, y) \in Y \times Y$. We call a geodesic bicombing $\{c_{xy}\}$ on Y γ-weakly convex, for some constant $\gamma \ge 1$, if each pair of geodesics $c_{xy} : [0, d(x, y)] \to Y$ and $c_{xz} : [0, d(x, z)] \to Y$ satisfy the inequality

$$d\left(c_{xy}(t\, d(x, y)), c_{xz}(t\, d(x, z))\right) \le \gamma\, t\, d(y, z) \tag{1}$$

for all $t \in [0, 1]$. A metric space which admits a γ-weakly convex geodesic bicombing is said to be a γ-weakly convex geodesic space.

Remark 2.2. On the one hand, the inequality (1) holds for $\gamma = 1$ on every geodesic space with convex distance function (for the unique geodesic bicombing), and on every normed vector space for the linear geodesic bicombing. On the other hand, one readily checks that a weakly convex geodesic space is Lipschitz n-connected for all $n \in \mathbb{N}$.

Proposition 2.3 generalizes a construction described in Section 1.5 of [1]. It proves that the metric space $\mathbf{Q}_Q(Y)$ is Lipschitz n-connected for all $n \in \mathbb{N}$ if Y is a weakly convex geodesic space. The Lipschitz connectedness of $\mathbf{Q}_Q(Y)$ is far from being obvious in this context since $\mathbf{Q}_Q(Y)$ does not necessarily inherit from Y a weakly convex geodesic bicombing. Indeed $\mathbf{Q}_2(\mathbb{R}^2)$ does not possess a weakly convex geodesic bicombing.

Proposition 2.3. Let Y be a γ-weakly convex geodesic space. Every Lipschitz multiple-valued function $f : \mathbf{S}^m \to \mathbf{Q}_Q(Y)$ extends to $F : \mathbf{B}^{m+1} \to \mathbf{Q}_Q(Y)$ with $\mathrm{Lip}(F) \le (\gamma + 8Q - 6)\,\mathrm{Lip}(f)$.

Proof. Set $D = 2\,\mathrm{Lip}(f) = \mathrm{diam}(\mathbf{S}^m)\,\mathrm{Lip}(f)$ and choose positive integers $s, Q_1, Q_2, \ldots, Q_s$ and points $p(i, k)$ in Y for $k = 1, \ldots, Q_i$, $i = 1, \ldots, s$ subject to the following requirements:

(1) $f(1, 0, 0, \ldots, 0) = \sum_{i=1}^{s} \sum_{k=1}^{Q_i} [\![p(i, k)]\!]$;

(2) If $i \ne j$ then $d(p(i, k), p(j, l)) > 4D$ for all $k \in \{1, \ldots, Q_i\}$, $l \in \{1, \ldots, Q_j\}$;

(3) For all $i \in \{1, \ldots, s\}$ and for all $k, l \in \{1, \ldots, Q_i\}$, there exists a sequence $k_1, k_2, \ldots, k_{Q_i}$ of not necessarily distinct elements of $\{1, \ldots, Q_i\}$ such that $k = k_1$, $l = k_{Q_i}$, and $d\left(p(i, k_j), p(i, k_{j+1})\right) \le 4D$ for $j = 1, \ldots, Q_i - 1$.

One notes that $\sum_{i=1}^{s} Q_i = Q$. We now define for each $i = 1, \ldots, s$,

$$f_i : \mathbf{S}^m \to \mathbf{Q}_{Q_i}(Y)$$

such that

$$\mathrm{spt}(f_i(x)) = \mathrm{spt}(f(x)) \cap \left(\cup_{k=1}^{Q_i} \mathbf{B}(p(i, k), D)\right)$$

for each $x \in \mathbf{S}^m$ where $\mathbf{B}(p(i, k), D)$ denotes the closed ball with center $p(i, k)$ and radius D. We will now check that the f_i are well defined. For $i \in \{1, \ldots, s\}$ and $x \in \mathbf{S}^m$, there is at least Q_i points in the support of $f_i(x)$ since

$$\mathcal{S}\left(f(x), \sum_{l=1}^{s} \sum_{k=1}^{Q_l} [\![p(l, k)]\!]\right) = \mathcal{S}(f(x), f(1, 0, \ldots, 0)) \le \mathrm{Lip}(f)\, |x - (1, 0, \ldots, 0)| \le D.$$

Suppose that there exists a point $p \in \mathrm{spt}(f_i(x)) \cap \mathrm{spt}(f_j(x))$ for $i, j \in \{1, \ldots, s\}$ such that $i \ne j$. Then, there exist $k \in \{1, \ldots, Q_i\}$, $l \in \{1, \ldots, Q_j\}$ with $d(p(i, k), p) \le D$ and $d(p(j, l), p) \le D$ hence $d(p(i, k), p(j, l)) \le 2D$ which contradicts (2). By these two observations, the f_i are well defined and $f = \sum_{i=1}^{s} f_i$. By construction, it is clear that $\mathcal{S}(f(x), f(y)) = \sum_{i=1}^{s} \mathcal{S}(f_i(x), f_i(y))$ for all $x, y \in \mathbf{S}^m$ hence $\mathrm{Lip}(f_i) \le \mathrm{Lip}(f)$ for $i = 1, \ldots, s$. By the definition of f_i and (3), we also notice that

$$\max\{\, d(p(i, 1), p) \;:\; p \in \;\mathrm{spt}(f_i(x))\} \le 4D(Q_i - 1) + D \le D(4Q - 3)$$

for $i = 1, \ldots, s$ and $x \in \mathbf{S}^m$. Let $\{c_{xy}\}$ be a γ-weakly convex geodesic bicombing on Y. We set

$$\theta : \mathbb{R}^{m+1} \backslash \{0\} \to \mathbf{S}^m : x \mapsto \tfrac{x}{|x|}$$

and observe that $\operatorname{Lip}(\theta | \{x \in \mathbb{R}^{m+1} \ : \ |x| = r\}) = 1/r$ for all $0 < r < \infty$. We can now define the extension

$$F : \mathbf{B}^{m+1} \to \mathbf{Q}_Q(Y)$$

$$F(0) = \textstyle\sum_{i=1}^s Q_i [\![p(i,1)]\!],$$

$$F(x) = \textstyle\sum_{i=1}^s \sum_{j=1}^{Q_i} \left[\!\!\left[c_{p(i,1), q_j^i(x)} \left(|x| \ d\left(p(i,1), q_j^i(x)\right)\right) \right]\!\!\right]$$

where we denote $f_i \circ \theta(x) = \sum_{j=1}^{Q_i} [\![q_j^i(x)]\!]$ for $i = 1, \ldots, s$ and for each $0 \neq x \in \mathbf{B}^{m+1}$. We easily check that $F|_{\mathbf{S}^m} = f$. Let $x, y \in \mathbf{B}^{m+1}$ such that $0 < |x| \leq |y|$ and fix $z = \frac{|x| y}{|y|}$. Since $|x| = |z|$, we see that $|y| - |z| = |y| - |x| \leq |y - x|$ and it is clear that $|x - z| \leq |x - y|$. It is also easy to check that $\theta(z) = \theta(y)$. On the one hand, we compute

$$\begin{aligned}
& \mathcal{S}(F(x), F(y)) \\
\leq \;& \mathcal{S}(F(x), F(z)) + \mathcal{S}(F(z), F(y)) \\
= \;& \mathcal{S}\left(\sum_{i=1}^s \sum_{j=1}^{Q_i} \left[\!\!\left[c_{p(i,1), q_j^i(x)} \left(|x| \ d(p(i,1), q_j^i(x))\right) \right]\!\!\right], \sum_{i=1}^s \sum_{j=1}^{Q_i} \left[\!\!\left[c_{p(i,1), q_j^i(z)} \left(|z| \ d(p(i,1), q_j^i(z))\right) \right]\!\!\right] \right) \\
& + \mathcal{S}\left(\sum_{i=1}^s \sum_{j=1}^{Q_i} \left[\!\!\left[c_{p(i,1), q_j^i(z)} \left(|z| \ d(p(i,1), q_j^i(z))\right) \right]\!\!\right], \sum_{i=1}^s \sum_{j=1}^{Q_i} \left[\!\!\left[c_{p(i,1), q_j^i(y)} \left(|y| \ d(p(i,1), q_j^i(y))\right) \right]\!\!\right] \right)
\end{aligned}$$

where we can suppose that

$$\mathcal{S}\left(f_i \circ \theta(x), f_i \circ \theta(z)\right) = \mathcal{S}\left(\sum_{j=1}^{Q_i} [\![q_j^i(x)]\!], \sum_{j=1}^{Q_i} [\![q_j^i(z)]\!] \right) = \max_{j=1,\ldots,Q_i} d\left(q_j^i(x), q_j^i(z)\right)$$

for $i = 1, \ldots, s$ and $q_j^i(z) = q_j^i(y)$ for $j = 1, \ldots, Q_i$ and $i = 1, \ldots, s$ since $\theta(z) = \theta(y)$. We conclude that

$$\begin{aligned}
& \mathcal{S}(F(x), F(y)) \\
\leq \;& \max_{i=1,\ldots,s} \max_{j=1,\ldots,Q_i} d\left(c_{p(i,1), q_j^i(x)} \left(|x| \ d(p(i,1), q_j^i(x))\right), c_{p(i,1), q_j^i(z)} \left(|z| \ d(p(i,1), q_j^i(z))\right) \right) \\
& + \max_{i=1,\ldots,s} \max_{j=1,\ldots,Q_i} d\left(c_{p(i,1), q_j^i(z)} \left(|z| \ d(p(i,1), q_j^i(z))\right), c_{p(i,1), q_j^i(y)} \left(|y| \ d(p(i,1), q_j^i(y))\right) \right) \\
\leq \;& \gamma \, |x| \max_{i=1,\ldots,s} \max_{j=1,\ldots,Q_i} d\left(q_j^i(x), q_j^i(z)\right) + (|y| - |z|) \max_{i=1,\ldots,s} \max_{j=1,\ldots,Q_i} d\left(p(i,1), q_j^i(y)\right) \\
\leq \;& \gamma \, |x| \max_{i=1,\ldots,s} \mathcal{S}(f_i \circ \theta(x), f_i \circ \theta(z)) + (|y| - |z|) \max_{i=1,\ldots,s} \max\{ \, d(p(i,1), p) \ : \ p \in \operatorname{spt}(f_i \circ \theta(y))\} \\
\leq \;& \gamma \, |x - z| \max_{i=1,\ldots,s} \operatorname{Lip}(f_i) + D(4Q - 3)|x - y|. \\
\leq \;& (\gamma + 8Q - 6) \operatorname{Lip}(f) |x - y|.
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
\mathcal{S}(F(x), F(0)) \;&=\; \mathcal{S}\left(\sum_{i=1}^s \sum_{j=1}^{Q_i} \left[\!\!\left[c_{p(i,1), q_j^i(x)} (|x| \ d(p(i,1), q_j^i(x)) \right]\!\!\right], \sum_{i=1}^s Q_i \, [\![p(i,1))]\!] \right) \\
&\leq\; |x| \max_{i=1,\ldots,s} \max\{ \, d(p(i,1), p) \ : \ p \in \operatorname{spt}(f_i \circ \theta(x))\} \\
&\leq\; (8Q - 6) \operatorname{Lip}(f) |x|.
\end{aligned}$$

$\square$

3. Nagata dimension of $\mathbf{Q}_Q(Y)$

We begin by giving the precise definition of the Nagata dimension. Suppose that (Y,d) is a metric space and $\mathcal{B} = (B_i)_{i\in I}$ is a family of subsets of Y. The family is called D-bounded, for some constant $D \geq 0$, if $\operatorname{diam}(B_i) = \sup\{ d(x,x') : x,x' \in B_i\} \leq D$ for all $i \in I$. For $s > 0$, the s-multiplicity of $\mathcal{B}$ is the infimum of all $n \geq 0$ such that every subsets of Y with diameter $\leq s$ meets at most n members of the family.

Definition 3.1. Let Y be a metric space. The Nagata dimension $\dim_N(Y)$ of Y is the infimum of all integers n with the following property: there exists a constant $c > 0$ such that for all $s > 0$, Y has a cs-bounded covering with s-multiplicity at most $n+1$.

The Section 2 in [8] gathers a number of basic properties of the Nagata dimension. We can quote for instance the following. The topological dimension of a metric space Y never exceeds $\dim_N(Y)$. The Nagata dimension is a bilipschitz invariant and, as it turns out, even a quasisymmetry invariant. The class of metric spaces with finite Nagata dimension includes all doubling spaces, metric ($\mathbb{R}-$)trees, Euclidean buildings, and homogeneous Hadamard manifolds, among others. Let us study the Nagata dimension of $\mathbf{Q}_Q(Y)$ in accordance with the Nagata dimension of Y.

Lemma 3.2. If Y is a metric space with $\dim_N(Y) = n < \infty$, then $\dim_N(\mathbf{Q}_Q(Y)) \leq (n+1)^Q - 1$.

Proof. Fix $s > 0$. Since $\dim_N(Y) = n$, there exists $c > 0$ such that Y admits a cs-bounded covering family $\mathcal{B} = (B_i)_{i\in I}$ with s-multiplicity at most $n+1$. For all $\sum_{j=1}^{Q}[\![i_j]\!] \in \mathbf{Q}_Q(I)$, we fix $B_{\sum_{j=1}^{Q}[\![i_j]\!]} := \mathbf{Q}_Q(Y) \cap \left\{\sum_{j=1}^{Q}[\![x_j]\!] \ : \ x_j \in B_{i_j} \text{ for } j = 1,\dots,Q\right\}$ and we define the collection

$$\mathcal{B}^* = \left(B_{\sum_{j=1}^{Q}[\![i_j]\!]}\right)_{\sum_{j=1}^{Q}[\![i_j]\!]\in\mathbf{Q}_Q(I)}.$$

One readily checks that B^* is a cs-bounded covering of $\mathbf{Q}_Q(Y)$. It remains to study the s-multiplicity of $\mathcal{B}^*$. Let A be a subset of $\mathbf{Q}_Q(Y)$ with $\operatorname{diam}(A) \leq s$. It is clear that there exist $A_1, A_2, \dots, A_Q \subset Y$ such that $A = \left\{\sum_{i=1}^{Q}[\![x_i]\!] \ : \ x_i \in A_i \text{ for } i = 1,\dots,Q\right\}$ with $\operatorname{diam}(A_i) \leq s$ for $i = 1,\dots,Q$. We know that $A_1,\dots,A_Q$ meet respectively at most $n+1$ members of $\mathcal{B}$. Consequently, A meets at most $(n+1)^Q$ members of $\mathcal{B}^*$. □

Corollary 3.3. If Y is a Banach space with a finite algebraic dimension, then the Nagata dimension of $\mathbf{Q}_Q(Y)$ is finite.

Proof. If Y has a finite algebraic dimension, the unit ball of Y is precompact. Consequently, Y is doubling and has a finite Nagata dimension hence $\dim_N(\mathbf{Q}_Q(Y)) < \infty$ by Lemma 3.2. □

We are now able to prove the results mentioned in the introduction. On the one hand, Theorem 1.1 immediately ensues from Lemma 2.1, Proposition 2.3 and Theorem 1.5 in [8]. On the other hand, Theorem 1.2 is an immediate consequence of Lemma 2.1, Proposition 2.3, Lemma 3.2, Corollary 3.3 and Corollary 1.8 in [8].

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